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1995 J. Phys. A: Math. Gen. 28 L637

(http://iopscience.iop.org/0305-4470/28/24/002)

J. Phys. A: Math. Gen. 28 (1995) L637–L638. Printed in the UK

LETTER TO THE EDITOR

Remarks on a recent theorem about conserved quantities

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Received 26 June 1995

Abstract. A recently proved theorem is used to derive a conserved quantity associated with a velocity-dependent symmetry for Lagrangian systems. In addition, a generalization of the theorem is given.

A theorem regarding conserved quantities for second-order dynamical systems has recently been given by Hojman [1], and subsequently generalized by González-Gascón [2], using geometric techniques; expressed in coordinates the result may be stated as follows. Let the functions $\eta_l(q, \dot{q}, t)$ determine a symmetry generator $E = \eta_l \frac{\partial}{\partial q_l} + \dot{\eta}_l \frac{\partial}{\partial \dot{q}_l}$ for the equations of motion $\ddot{q}_l = \alpha_l(q, \dot{q}, t)$; if a function $\lambda(q, \dot{q}, t)$ can be found such that the quantity

$$\Omega = \frac{\partial \alpha_l}{\partial \dot{q}_l} + \frac{\mathrm{d}}{\mathrm{d}t} \ln \lambda$$

vanishes, then a constant of the motion is given by

$$\phi = E(\ln \lambda) + \frac{\partial \eta_l}{\partial q_l} + \frac{\partial \dot{\eta}_l}{\partial \dot{q}_l} \qquad \dot{\phi} = 0.$$

(We use the overdot to denote the total time derivative along a trajectory, $\dot{\phi} \equiv \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial q_l}\dot{q}_l + \frac{\partial \phi}{\partial \dot{q}_l}\alpha_l$.)

In this letter we apply the theorem to derive a certain conserved quantity for Lagrangian systems; this conserved quantity is associated with the presence of a velocity-dependent (non-point) symmetry. In addition, we generalize the theorem by showing that it is sufficient for Ω to be an invariant of the symmetry group in order for ϕ to be conserved.

The theorem referred to above holds whether or not the equations of motion are derivable from a Lagrangian. However, in the event that there *does* exist a Lagrangian $L(q, \dot{q}, t)$ for the system $\ddot{q}_l = \alpha_l$, it can be shown quite generally that

$$\frac{\partial \alpha_l}{\partial \dot{q}_l} + \frac{\mathrm{d}}{\mathrm{d}t}(\ln D) = 0$$

where D is the determinant of the matrix whose elements are $\partial^2 L/\partial \dot{q}_k \partial \dot{q}_l$. We may choose $\lambda = D$, and therefore the condition $\Omega = 0$ can *always* be satisfied for a Lagrangian system, leading automatically to the conserved quantity

$$\phi = \frac{\partial \eta_l}{\partial q_l} + \frac{\partial \dot{\eta}_l}{\partial \dot{q}_l} + E\{\ln D\}.$$

The particular interest of this result lies in the fact that the η_l may depend on the velocities $\dot{q}_l$; thus ϕ is associated with a *non-point* symmetry. Should the η_l be independent of the velocities (in which case they determine a *point* symmetry) we find that

$$\frac{\partial \dot{\eta}_l}{\partial \dot{q}_l} = \frac{\partial \eta_l}{\partial q_l}$$

and the conserved quantity reduces to

$$\psi = 2\frac{\partial \eta_l}{\partial q_l} + E\{\ln D\}.$$

This conserved quantity, related to point symmetries, is well known [3], and has been derived using the fact that if E is a point symmetry, then the Lagrangian $L' = E(L)$ leads to the same equations of motion as L. However, the conserved quantity ϕ cannot be derived in this way, since it is not generally true that $L' = E(L)$ and L are equivalent Lagrangians if E is a velocity-dependent symmetry.

Finally, we prove a generalization of the theorem of [2] by showing that it is sufficient that Ω be an invariant of the symmetry group in order for ϕ to be conserved; that is, we show that $E\{\Omega\} = 0$ implies $\dot{\phi} = 0$. This is accomplished by explicitly differentiating ϕ to demonstrate that $\dot{\phi} = E\{\Omega\}$. The calculation is greatly simplified by the following easily proven identifies ($f(q, \dot{q}, t)$ is arbitrary):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial f}{\partial \dot{q}_l}\right\} = \frac{\partial \dot{f}}{\partial \dot{q}_l} - \frac{\partial f}{\partial q_l} - \frac{\partial f}{\partial \dot{q}_m}\frac{\partial \alpha_m}{\partial \dot{q}_l} \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial f}{\partial q_l}\right\} = \frac{\partial \dot{f}}{\partial q_l} - \frac{\partial f}{\partial \dot{q}_m}\frac{\partial \alpha_m}{\partial q_l} \tag{2}$$

$$E\left\{\frac{\partial f}{\partial \dot{q}_l}\right\} = \frac{\partial}{\partial \dot{q}_l}E\{f\} - \frac{\partial \eta_k}{\partial \dot{q}_l}\frac{\partial f}{\partial q_k} - \frac{\partial \dot{\eta}_k}{\partial \dot{q}_l}\frac{\partial f}{\partial \dot{q}_k}. \tag{3}$$

The condition that the η_l define a symmetry [4] for $\ddot{q}_l = \alpha_l$ is given by $\ddot{\eta}_l = E\{\alpha_l\}$; using this we can show that if E is a symmetry, then

$$\frac{\mathrm{d}}{\mathrm{d}t}E\{f\} = E\{\dot{f}\}. \tag{4}$$

Differentiating ϕ we obtain

$$\dot{\phi} = E\left\{\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda\right\} + \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial \eta_l}{\partial q_l} + \frac{\partial \dot{\eta}_l}{\partial \dot{q}_l}\right\} \tag{5}$$

where we have used (4).

Using (1) and (2) we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial \eta_l}{\partial q_l} + \frac{\partial \dot{\eta}_l}{\partial \dot{q}_l}\right\} = \frac{\partial \ddot{\eta}_l}{\partial \dot{q}_l} - \frac{\partial \dot{\eta}_s}{\partial \dot{q}_m}\frac{\partial \alpha_m}{\partial \dot{q}_s} - \frac{\partial \eta_s}{\partial \dot{q}_m}\frac{\partial \alpha_m}{\partial q_s}. \tag{6}$$

Furthermore, equation (3) allows us to show that $E\{\frac{\partial \alpha_l}{\partial \dot{q}_l}\}$ equals the right-hand side of (6), so that we may put

$$E\left\{\frac{\partial \alpha_l}{\partial \dot{q}_l}\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial \eta_l}{\partial q_l} + \frac{\partial \dot{\eta}_l}{\partial \dot{q}_l}\right\}. \tag{7}$$

Using (7) in (5) then yields

$$\dot{\phi} = E\left\{\frac{\partial \alpha_l}{\partial \dot{q}_l} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda\right\}$$

which is the desired result.

References

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